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A Restrictive Padé approximation for the solution of RLW equation

Research Article

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Abstract: Solving RLW equation numerically has many difficulties for accuracy. Restrictive Padé (RP) approximation is used. The numerical solution of RLW equation by RP scheme leads to accurate and efficient results. The stability analysis is discussed. Numerical results are presented.

MSC: 66M99 • 35Q51

Keywords: Partial differential equation (PDE) • Regularized long wave (RLW) • Restrictive Padé approximation

1. Introduction

The generalized regularized long-wave (GRLW) equation take the form

$$u_t + u_x + \alpha\, u^p u_x - \mu\, u_{xxt} = 0 \tag{1}$$

Where α and μ are positive constants, $p \geq 1$ is an integer. And the subscripts symbols mark the partial derivatives of x, t variables. One of the special cases for this equation when $p = 1$, the Eq. (1) became

$$u_t + u_x + \alpha\, u u_x - \mu\, u_{xxt} = 0 \tag{2}$$

Which called as the regularized long-wave (RLW) equation which particularly describes the behavior of the undular bore [1] and to describe nonlinear dispersive waves. Different numerical techniques are used to solve this equation. Some of these methods are finite difference method [2], Galerkin method with extrapolation techniques [3], The Tanh function method [4], finite element methods including collocation method with quadratic B-splines [5]which also used to solve Rosenau-KdV Eq. [6], cubic B-splines [7], cubic splines in tension [8], a stable spectral collocation method [9], Haar wavelet method [10] and final Adomain Decomposition method [11].

Restrictive approximation is a new technique developed by Ismail et al [12], there are two type of restrictive approximation, Restrictive Taylor approximation (RT) which used by Rageh et al to solve Gardner and KdV Eqs. [13], and restrictive Padé approximation (RPA), which used for solving parabolic PDE [14]-[16], RPA also used by Ismail et al to solve hyperbolic PDE [17]-[19] and developed by Ismail et al to solve Schrodinger Eq. [20][21] and finally Ismail et al used RPA to solve generalized Fisher and Burger Fisher Eq. [22][23]. It yields more accurate results, in this paper, we propose to use the restrictive Padé approximation scheme to solve the RLW Eq. (2) numerically and compare our results with other numerical methods, and Numerical example shows that the present scheme results are more accurate when it compared with the exact solution, Conclusions and analysis are presented.

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2. Restrictive Padé approximation

The restrictive Padé approximation of the function f(x) can be written, as done in [14], in the form,

$$RPA[M+\alpha/N]_{f(x)} = \frac{\sum_{i=0}^{M} a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{M+i}}{1+\sum_{i=1}^{N} b_i x^i} \tag{3}$$

Where α is the positive integer does not exceed the degree of the numerator N i.e., $\alpha = 0\ (1)$N,

$$f(x) - RPA[M+\alpha/N]_{f(x)} = O(x^{M+N+1}) \tag{4}$$

The unknowns a_i, b_i and ε_i are to be determined such that

$$RPA[M+\alpha/N]_{f(x_i)}(x) = f(x_i)\ i = 1(1)k$$

The restrictive Padé approximation RPA[1/1] of the exponential matrix e^{rA} is given by [14]

$$RPA[1/1]_{exp(rA)}(r) = (I + (\varepsilon - \frac{1}{2}A)r)^{-1}(I + (\varepsilon + \frac{1}{2}A)r) \tag{5}$$

Where A is $n \times n$ real constant matrix, and ε is the diagonal matrix $\varepsilon = [\varepsilon_{i,j}]$: $\varepsilon_{i,i} = \varepsilon_i$, $\varepsilon_{i,j} = 0$ otherwise i,j = 1(1)n

3. Analytic solution of RLW

The exact solution of RLW Eq.(2) is [24]

$$u(x,t) = 3c\,Sech^2(P(x - vt - x_o)) \tag{6}$$

This is the solution of a single solitary wave with amplitude $3c$, width $P = \sqrt{\frac{c}{4\mu(c+1)}}$ initially centered at x_o and $v = 1+c$ is the wave velocity.

4. Restrictive Padé schemes for the RLW equation

To solve RLW Eq. (2) with initial values

$$u(x,0) = 3c\,Sech^2(P(x - x_o)) \tag{7}$$

Using restrictive Padé approximation as used in [14]-[23], where subscripts x and t denote the differentiation and $u \longrightarrow 0$ as $x \longrightarrow \pm\infty$. In numerical applications, we use periodic boundary conditions for a region $a \le x \le b$.

$$\begin{aligned}
(u_t)_i^j &= \frac{u_i^{j+1} - u_i^j}{k} \\
((1+u)\,u_x)_i^j &= \frac{1}{2}(1+u_i^j)\frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + \frac{1}{2}(1+u_i^j)\frac{u_{i+1}^{j} - u_{i-1}^{j}}{2h} \\
(u_{xxt})_i^j &= \frac{(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}) - (u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j})}{kh^2}
\end{aligned} \tag{8}$$

Here, we have used the forward-difference operator in time t and the central difference operator in space x. Then the scheme for the RLW equation became

$$\begin{aligned}
&(\mu r + \frac{khr}{4}(1+(u_i^j))u_{i-1}^{j+1} - (2\mu r + 1 + r\varepsilon_i)u_i^{j+1} + (\mu r - \frac{khr}{4}(1+(u_i^j))u_{i+1}^{j+1} \\
&= (\mu r - \frac{khr}{4}(1+(u_i^j))u_{i-1}^{j} - (2\mu r + 1 + r\varepsilon_i)u_i^{j} + (\mu r + \frac{khr}{4}(1+(u_i^j))u_{i+1}^{j}
\end{aligned} \tag{9}$$

where i and j are nonnegative integers, $r = \frac{1}{h^2}$ To determine the restrictive term ε_i, we need to know an additional condition u(x,k), i.e. ε_i must be given such that the error at certain r is zero after which, we should use the Eq. (9) for another calculations to get the required solution.

5. Stability Analysis

The Von Neumann technique is applied to check the stability of the RP scheme Eq. (9), we must linearize the nonlinear term of the RLW equation to perform Von Neumann method by assuming that the quantity u in nonlinear term uu_x as locally constant [3]. then assume the numerical solution can be expressed by means of a Fourier series

$$u_i^j = \xi^j e^{IKih} \tag{10}$$

where $I = \sqrt{-1}$,k is the mode number and h is the element size, substitute by (10) on (9) and let $Q = \frac{khr}{4}(1 + u_i^j)$ then

$$\xi = \frac{2\mu r Cos(kh) - 2\mu r - 1 - r\varepsilon_i + I2QSin(kh)}{2\mu r Cos(kh) - 2\mu r - 1 - r\varepsilon_i - I2QSin(kh)} \tag{11}$$

Or

$$\xi = \frac{A + IB}{A - IB} \tag{12}$$

where

$$A = 2\mu r Cos(kh) - 2\mu r - 1 - r\varepsilon_i,\ B = 2QSin(kh)$$

thus $|\xi| = 1$, so the RP scheme is unconditionally stable.

6. Numerical Results

Numerical solutions of RLW equation are obtained for a motion of a single solitary wave with Maxwellian initial condition. Absolute error is used to show how good the numerical results in comparison with the exact results. Consider Eq. (2) with boundary conditions

$$u(a,t) = u(b,t) = 0$$

and the initial condition (7).

6.1. Example (1)

For the purpose of comparing with the earlier work [27], which uses highly accurate Modified Laplace Adomian Decomposition method to solve the same problem with the following parameters $c = 0.1$, $\mu = 1$, $\alpha = 1$,$\Delta x = 0.2$, $x_o = 0$ and $\Delta t = 0.01$ over the interval [-5, 5]. The exact solution of this problem is given by (6). We use our present method Restrictive Padé (RP) to solve this equation numerically, And finding the error at various times up to $\Delta t = 0.05$. using the exact solution reported data in Table 1, shows the absolute error for Modified Laplace Adomian decomposition method [27] and present method, Fig. 1(a) shows the graph of the exact solution, Fig. 1(b) shows the numerical solution using our present method and Fig. 1(c) shows exact and numerical solution at the same graph.

6.2. Example (2)

Consider Eq. (2) with the initial condition (7) the exact solution of this problem is given by (6). For another comparison with the earlier work [28], which uses variational iteration method to solve RLW, We use the Restrictive PadÃĽ (RP) method, all computations are done for the same parameters considered in [28] as following $c = 1$, $\mu = 1$, $\alpha = 1$, $\Delta x = 0.125$, $x_o = 0$ and $\Delta t = 0.001$ over the interval [-10, 10]. Using the exact solution reported data in Table 2 shows comparison between absolute error for VIM [28] and present method RP. Also Fig. 2(a) shows the graph of the exact solution, Fig. 2(b) shows the numerical solution using our present method and Fig. 2(c) shows exact and numerical solution at the same graph.

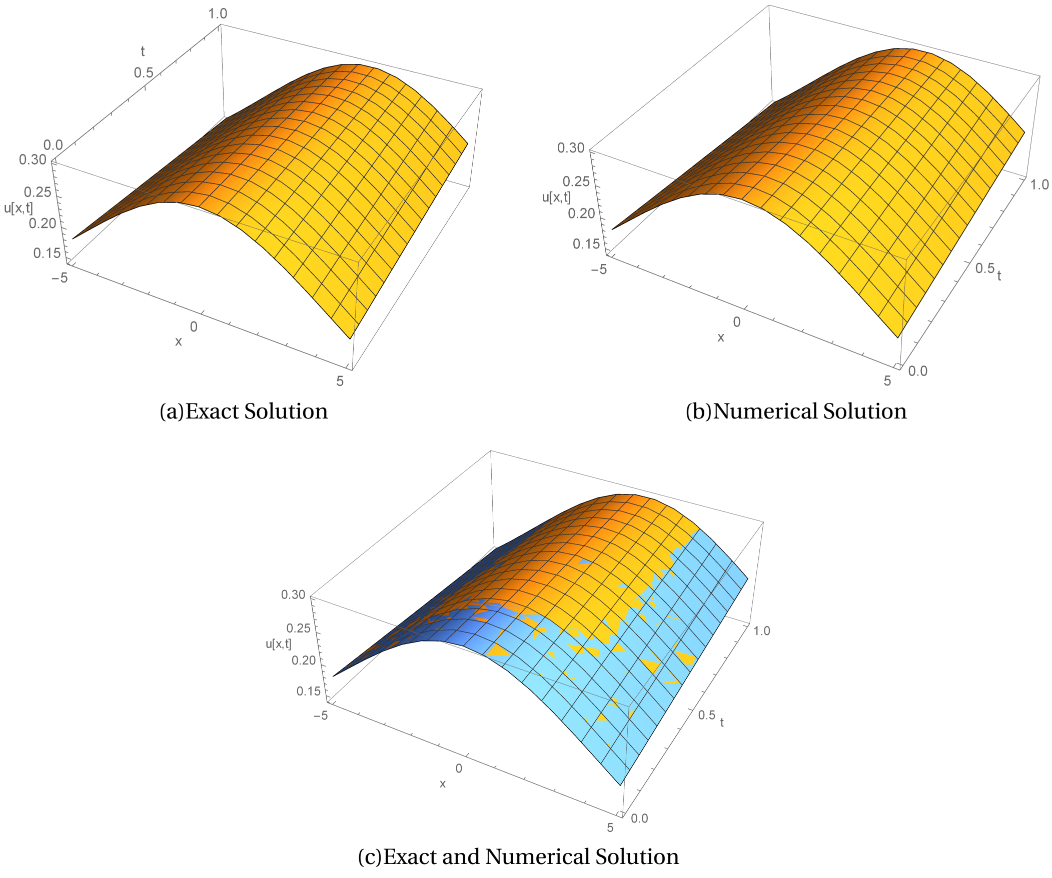
6.3. Example (3)

The accuracy of the method is measured in this example by using the error norms L_2 and L_∞. We also examined our results by calculating the following three conserved quantities corresponding to mass, momentum and energy, for the periodic boundary conditions, they are [25]

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u dx \cong \Delta x \sum_{i=1}^{N} U_{i,j} \\ I_2 &= \int_{-\infty}^{\infty} [u^2 + \mu u_x^2] dx \cong \Delta x \sum_{i=1}^{N} [(U_{i,j})^2 + \mu((U_x)_{i,j})^2] \\ I_3 &= \int_{-\infty}^{\infty} [u^3 + 3u^2] dx \cong \Delta x \sum_{i=1}^{N} [(U_{i,j})^3 + 3((U)_{i,j})^2] \end{aligned} \tag{13}$$

Table 1. Comparison between absolute error for highly accurate Modified Laplace Adomian Decomposition method [27] and present method RP

x	t = 0.02		t = 0.03	
	[27] method	R.P method	[27] method	R.P method
0.1	4.291×10^{-07}	3.121×10^{-11}	9.145×10^{-08}	9.588×10^{-11}
0.2	1.811×10^{-06}	2.482×10^{-11}	1.976×10^{-06}	7.660×10^{-11}
0.3	3.144×10^{-06}	1.887×10^{-11}	3.965×10^{-06}	5.863×10^{-11}
0.4	4.409×10^{-06}	1.339×10^{-11}	5.845×10^{-06}	4.207×10^{-11}
0.5	5.582×10^{-06}	8.415×10^{-12}	7.583×10^{-06}	2.703×10^{-11}
x	t = 0.04		t = 0.05	
	[27] method	R.P method	[27] method	R.P method
0.1	1.100×10^{-06}	1.963×10^{-10}	2.593×10^{-06}	3.347×10^{-10}
0.2	1.651×10^{-06}	1.575×10^{-10}	8.412×10^{-07}	2.697×10^{-10}
0.3	4.290×10^{-06}	1.213×10^{-10}	4.126×10^{-06}	2.091×10^{-10}
0.4	6.776×10^{-06}	8.799×10^{-11}	7.210×10^{-06}	1.531×10^{-10}
0.5	9.067×10^{-06}	5.765×10^{-11}	1.004×10^{-05}	1.021×10^{-10}



(a)Exact Solution (b)Numerical Solution

(c)Exact and Numerical Solution

Fig. 1. Single solitary wave at $c = 0.1$, $\mu = 1 and\ x_o = 0$

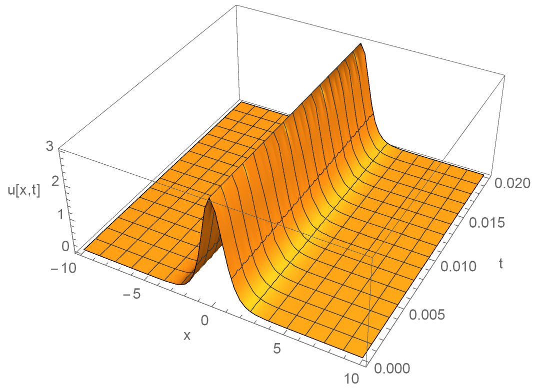
The analytical values of conservation quantities can be found as [26]

$$I_1 = \frac{6c}{p}, I_2 = \frac{12c^2}{p} + \frac{48pc^2\mu}{5}, I_3 = \frac{36c^2}{p}(1 + \frac{4c}{5}) \tag{14}$$

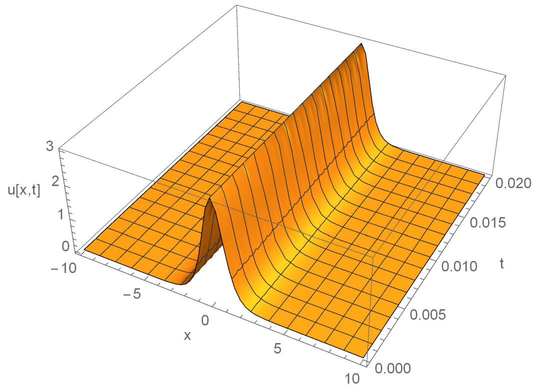
To allow comparison with the previous method parameters are taken as $\mu = 1$ and $\alpha = 1$ The analytical invariants for $c = 0.03$ found using Eq. (14) are $I_1 = 2.109407$, $I_2 = 0.127302$ and $I_3 = 0.388806$. Table 3 displays invariants and Table 4 shows comparison between error norms for fully implicit method [28] and the present RP method at $c = 0.03$, $x_o = 0$, the space step $h = 0.1$ and the time step $k = 0.2$ through the interval $-40 \leq x \leq 60$, The invariants and error norms of the proposed scheme are given for times up to $t = 20$.

Table 2. Comparison between absolute error for Variational iteration method [28] and present method RP

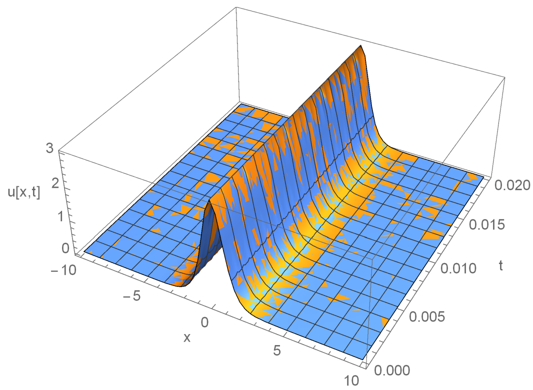
x	t = 0.002		t = 0.003		t = 0.004	
	[28] method	R.P method	[28] method	R.P method	[28] method	R.P method
-5	1.07×10^{-06}	5.614×10^{-12}	1.60×10^{-06}	1.671×10^{-11}	2.12×10^{-06}	3.317×10^{-11}
-2.5	9.55×10^{-05}	5.445×10^{-09}	1.42×10^{-04}	1.626×10^{-08}	1.88×10^{-04}	3.237×10^{-8}
0	1.44×10^{-04}	6.068×10^{-08}	3.24×10^{-04}	1.821×10^{-07}	1.88×10^{-04}	3.237×10^{-08}
2.5	9.88×10^{-05}	6.245×10^{-09}	1.49×10^{-04}	1.882×10^{-08}	1.49×10^{-04}	3.781×10^{-08}
5	1.10×10^{-06}	6.541×10^{-12}	1.66×10^{-06}	1.977×10^{-11}	2.23×10^{-06}	3.985×10^{-11}
x	t = 0.005		t = 0.01			
	[28] method	R.P method	[28] method	R.P method		
-5	1.60×10^{-06}	5.485×10^{-11}	5.11×10^{-06}	2.374×10^{-10}		
-2.5	2.33×10^{-04}	5.370×10^{-08}	4.46×10^{-04}	2.361×10^{-07}		
0	9.00×10^{-04}	6.080×10^{-07}	3.60×10^{-03}	2.746×10^{-06}		
2.5	2.53×10^{-04}	6.331×10^{-08}	5.27×10^{-04}	2.914×10^{-07}		
5	2.80×10^{-06}	6.692×10^{-11}	5.76×10^{-06}	3.127×10^{-10}		



(a)Exact Solution



(b)Numerical Solution



(c)Exact and Numerical Solution

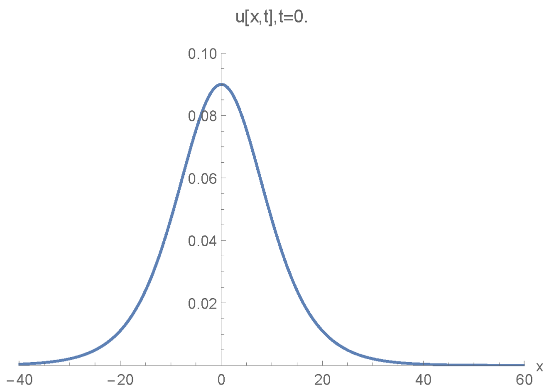
Fig. 2. Single solitary wave at $c = 1$, $\mu = 1 and\ x_o = 0$

Table 3. Invariants for Example (3) (section 6.3) which represent the single solitary wave for $h = 0.1, k = 0.2, -40 \le x \le 60$ *and* $c = 0.03$

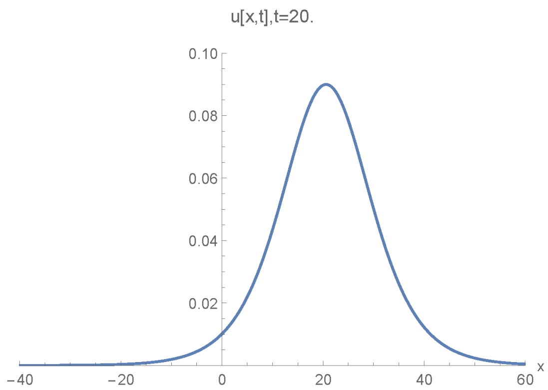
t	I_1	I_2	I_3
0	2.107066878774	0.1273012562693	0.388804675401
2	2.107697420565	0.1273004628316	0.388802178601
4	2.108088350288	0.1272962472639	0.388789122315
6	2.108283914748	0.1272874985994	0.388762058160
8	2.108313268588	0.1272738623070	0.388719874819
10	2.108194866806	0.1272557444351	0.388663802762
12	2.107937977532	0.1272341185227	0.388596805159
14	2.107541895132	0.1272102385609	0.388522667787
16	2.106993350274	0.1271853688370	0.388445118584
18	2.106262376474	0.1271606080459	0.388367192712
20	2.105296544298	0.1271368403901	0.388290897941

Table 4. Comparison between error norms for fully implicit method [29] and the present RP method

t	L_2		L_∞	
	R.P method	[29] method	R.P method	[29] method
0	0	0	0	0
2	8.219×10^{-6}	0.000070	8.83306×10^{-7}	0.000074
4	0.000037	0.000150	4.10541×10^{-6}	0.000123
6	0.000090	0.000237	0.000010	0.000152
8	0.000167	0.000323	0.000019	0.000166
10	0.00026	0.000401	0.000029	0.000174
12	0.00039	0.000468	0.000042	0.000179
14	0.00052	0.000524	0.000055	0.000182
16	0.00065	0.000570	0.000069	0.000184
18	0.00079	0.000608	0.000081	0.000186
20	0.00093	0.000642	0.000092	0.000233



(a)Single solitary wave at $t = 0$



(b)Single solitary wave at $t = 20$

Fig. 3. Single solitary wave

7. Conclusion

After solving Examples (1) (section 6.1) and (2) (section 6.2), Table 1 and Table 2 shows comparison between the absolute error of the considered Restrictive Padé (RP) approximation and highly accurate Modified Laplace Adomian Decomposition method (ADM) which used to solve Example (1) (section 6.1) [27] and variational iteration method which used to solve Example (2) (section 6.2) [28], Also as shown in Example (3) (section 6.3), the change in invariants is less than 10^{-3} and the comparison in Table 4 shows that the norms of error result from present RP method are less than that we get from Fully implicit method [29]. The results prove that the present method is more accurate than the previously used methods, i.e. the global error for RP method is less by at least 10^{-3} than the previous method.

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